

A Stern Tri-atomic sequence (Pascal's Pyramid with Memory) for a Multi-dimensional Continued Fraction Algorithm

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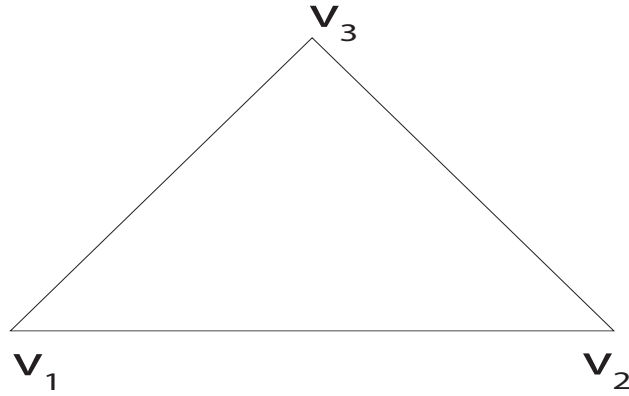
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Abstract

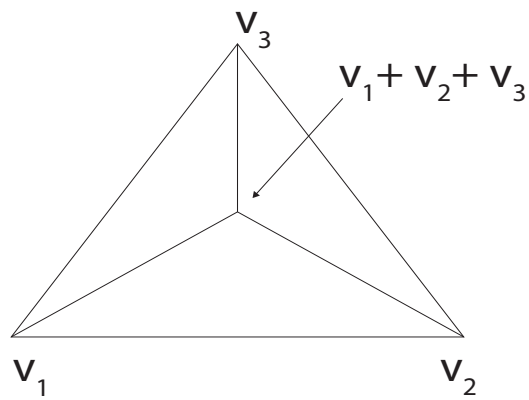
Continued fractions are linked to Stern's diatomic sequence $0, 1, 1, 2, 1, 3, 2, 3, 1, 4, \dots$ (given by the recursion relation $A_{2n} = A_n$ and $A_{2n+1} = a_n + a_{n+1}$, where $A_0 = 0$ and $A_1 = 1$), which has long been known. Using a particular multi-dimensional continued fraction algorithm (the Farey algorithm), we will generalize the diatomic sequence to a collection of numbers that quite naturally should be called the tri-atomic sequence (or a two-dimensional Pascal with memory sequence). As continued fractions and the diatomic sequence can be thought of as coming from systematic subdivisions of the unit interval, this new tri-atomic sequence will arise by a systematic subdivision of a triangle. We will discuss some of the algebraic properties for the tri-atomic sequence.

1 Introduction

Start with a triangle \triangle

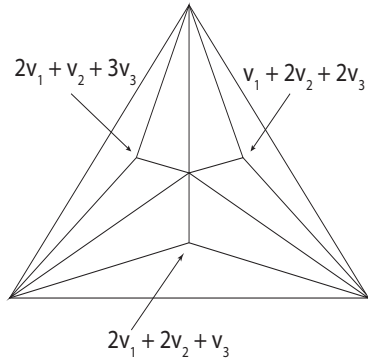


with vertices v_1, v_2, v_3 . In this paper we will consider both the case when the vertices denote numbers and when the vertices denote vectors. From \triangle form three new triangles



where $\triangle(0)$ has vertices $v_1, v_2, v_1 + v_2 + v_3$, $\triangle(1)$ has vertices $v_2, v_3, v_1 + v_2 + v_3$

and $\triangle(2)$ has vertices $v_3, v_1, v_1 + v_2 + v_3$. Each of these triangles can be divided into three more triangles, giving us nine triangles



We will identify each of these triangles with their vertices. Thus we start with our initial triangle (which we will say is at level 0):

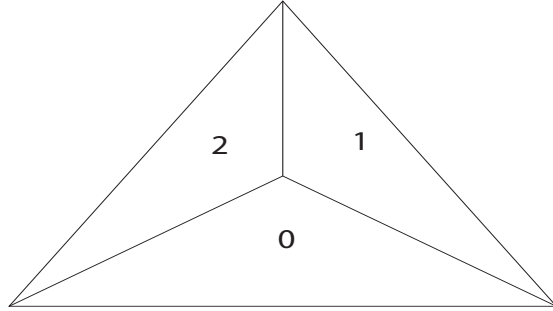
$$\triangle = (v_1, v_2, v_3).$$

At level one we have

$$\triangle(0) = (v_1, v_2, v_1 + v_2 + v_3)$$

$$\triangle(1) = (v_2, v_3, v_1 + v_2 + v_3)$$

$$\triangle(2) = (v_3, v_1, v_1 + v_2 + v_3)$$



while for level two we have

$$\triangle(00) = (v_1, v_2, 2v_1 + 2v_2 + v_3)$$

$$\triangle(01) = (v_2, v_1 + v_2 + v_3, 2v_1 + 2v_2 + v_3)$$

$$\triangle(02) = (v_1 + v_2 + v_3, v_1, 2v_1 + 2v_2 + v_3)$$

$$\triangle(10) = (v_2, v_3, v_1 + 2v_2 + 2v_3)$$

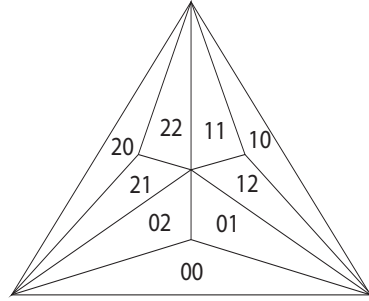
$$\triangle(11) = (v_3, v_1 + v_2 + v_3, v_1 + 2v_2 + 2v_3)$$

$$\triangle(12) = (v_1 + v_2 + v_3, v_2, v_1 + 2v_2 + 2v_3)$$

$$\triangle(20) = (v_3, v_1, 2v_1 + v_2 + 2v_3)$$

$$\triangle(21) = (v_1, v_1 + v_2 + v_3, 2v_1 + v_2 + 2v_3)$$

$$\triangle(21) = (v_1 + v_2 + v_3, v_3, 2v_1 + v_2 + 2v_3).$$



Continuing this process produces a sequence of triples

$$\triangle, \triangle(0), \triangle(1), \triangle(2), \triangle(00), \triangle(01), \triangle(02), \triangle(10), \dots$$

The goal of this paper is to study some of the properties of this sequence. As we will see, this sequence can quite naturally be called the *Stern's Tri-atomic sequence*, since it is an analog for the classical Stern diatomic sequences (or Pascal with memory), which we quickly review in section two.

In section three we give a more formal presentation and definitions for Stern tri-atomic sequence. Section four shows how Stern's tri-atomic sequence arises from the multi-dimensional continued fraction stemming from the Farey map, in analog to the link between Stern's diatomic sequence and classical continued fractions. Sections five, six and seven discuss some of the properties for this sequence. Many of this properties are analogs of corresponding properties for Stern's diatomic sequence.. We will conclude with open questions.

2 The Classical Stern Di-atomic Sequence

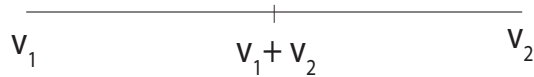
We now quickly review the basics of Stern's diatomic sequence. A recent Monthly article by Northshield [12] gives a good overview of this sequence and, more importantly for this paper, its relation to continued fractions. The goal of this paper is to take a particular generalization of continued fractions and find the corresponding generalization of Stern's diatomic sequences. Lehmer [11] lists a number of properties of this sequence. Part of the goal for this current paper will be in showing how these properties usually generalize for our new Stern triatomic sequences.

In the last section, we systematically subdivided a triangle into smaller triangles. As a triangle is a two-dimensional simplex, it is natural to find the analog for one-dimensional simplices, namely intervals.

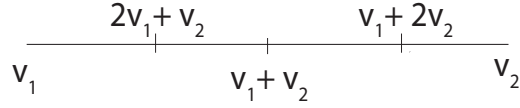
Consider an interval, with endpoints v_1 and v_2 :



Then subdivide:



We can continue, getting:



The traditional Stern diatomic sequence is doing this for the case when $v_1 = v_2 = 1$.

More precisely, Stern's diatomic sequence is given by the recursion formulas

$$\alpha_0 = 0, \alpha_1 = 1$$

and

$$\alpha_{2n} = \alpha_n$$

$$\alpha_{2n+1} = \alpha_n + \alpha_{n+1}$$

The first few terms are

$$0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 4, \dots$$

It is standard to write this as

$$\begin{array}{cccccccccc} & \alpha_1 & & & & & & & & \alpha_2 \\ \alpha_2 & & & & & \alpha_3 & & & & \alpha_4 \\ \alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 & & \alpha_8 \\ \alpha_8 & \alpha_9 & \alpha_{10} & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \end{array}$$

For $\alpha_1 = 1$, we get

$$\begin{array}{cccccccccccccccc} 1 & & & & & & & & & & & & & & 1 \\ 1 & & & & & 2 & & & & & & & & & 1 \\ 1 & & & 3 & & 2 & & & 3 & & & & & & 1 \\ 1 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & & & & & & & 1 \\ 1 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 2 & 7 & 5 & 8 & 3 & 7 & 4 & 5 & 1 \end{array}$$

Thus to create the next level, we simply interlace the original level with the sum of adjacent entries. This is why this sequence is called by Knauf [33]

Pascal with Memory. (It was Knauf's work on linking this sequence with statistical mechanics, which is also in [32] [34] [35] [36] [37], that led the author to look at these sequences. Other work linking statistical mechanics with Stern's diatomic sequences, though these works do not use the rhetoric of Stern's diatomic sequences, include [22] [23] [25] [26] [27] [28] [31] [32] [39] [40] [41] [42] [43] [44])

A matrix approach for the Stern diatomic sequence $\alpha_1, \alpha_2, \dots$ is as follows. Set

$$b_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $N > 2$ be a positive integer. Then there is a tuple $(k : i_k, \dots, i_1)_S$ with k a positive integer and each $i_j \in \{0, 1\}$ such that

$$N = 2^k + 1 + i_k 2^{k-1} + \dots + i_1.$$

(The subscript S stands for 'Stern'.) For example $3 = 2^1 + 1 + 0 = (1 : 0)_S$, $4 = 2^1 + 1 + 1 = (1 : 1)_S$, $5 = 2^2 + 1 + 0 \cdot 2^1 + 0 = (2 : 0, 0)_S$ and $16 = 2^3 + 1 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 = (3 : 1, 1, 1)_S$ Then we have

$$\alpha_N = \begin{pmatrix} 0 & 1 \end{pmatrix} M b_{i_k} \cdots b_{i_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is just systemizing the subdivision of the unit interval via Farey fractions.

3 A Stern Tri-atomic sequence

We will write down a sequence that captures all of the terms for our Stern tri-atomic sequence (whose rigorous definition will be given in a moment) by creating a sequence of triples. The initial triple is

$$\Delta = (v_1, v_2, v_3) = (1, 1, 1).$$

We are using the notation Δ to suggest the initial triangle in the introduction.

For any index

$$I = (i_1, i_2, \dots, i_n),$$

with each i_k being zero, one or two, we will associate a triple which we denote by $\Delta(I)$.

Suppose we know the triple

$$\Delta(I) = (v_1(I), v_2(I), v_3(I)).$$

Consider the three matrices:

$$A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Then we set

$$\begin{aligned} \Delta(I, 0) &= \Delta(I)A_0 = (v_1(I), v_2(I), v_3(I))A_0 \\ \Delta(I, 1) &= \Delta(I)A_1 = (v_1(I), v_2(I), v_3(I))A_1 \\ \Delta(I, 2) &= \Delta(I)A_2 = (v_1(I), v_2(I), v_3(I))A_2. \end{aligned}$$

Thus we have

$$\begin{aligned} v_1(I, 0) &= v_1(I), & v_2(I, 0) &= v_2(I), & v_3(I, 0) &= v_1(I) + v_2(I) + v_3(I) \\ v_1(I, 1) &= v_2(I), & v_2(I, 1) &= v_3(I), & v_3(I, 1) &= v_1(I) + v_2(I) + v_3(I) \\ v_1(I, 2) &= v_3(I), & v_2(I, 2) &= v_1(I), & v_3(I, 2) &= v_1(I) + v_2(I) + v_3(I). \end{aligned}$$

For $I = (i_1, \dots, i_n)$ and $J = (j_1, \dots, j_m)$, we say that $I < J$ if $n < m$ or, when $n = m$, there is an integer k such that $1 \leq k \leq n$, for all integers $1 \leq l < k$ we have $i_l = j_l$ and $i_k < j_k$. Thus we have the ordering

$$(0) < (1) < (2) < (00) < (01) < (02) < (10) < (11) < (12) < (20) < (21) < (22) < (000) < \dots$$

Definition 3.1. *The Stern Tri-atomic sequence is*

$$\Delta, \Delta(0), \Delta(1), \Delta(2), \Delta(00), \Delta(01), \Delta(02), \Delta(10), \dots$$

and hence the sequence formed from all of the triples $\Delta(I)$ using the above ordering on the indices I .

Thus the first terms are

$$1, 1, 1, 1, 1, 3, 1, 1, 3, 1, 1, 3, 1, 1, 5, 1, 3, 5, 3, 1, 5, 1, 1, 5, 1, 3, 5, 3, 1, 5, 1, 1, 5, 1, 3, 5, 3, 1, 5$$

Definition 3.2. *A triple $\Delta(i_1, \dots, i_n)$ is said to be in level n .*

Thus the level 1 triples are

$$\Delta(0), \Delta(1), \Delta(2) = 1, 1, 3, 1, 1, 3, 1, 1, 3$$

while the level 2 triples are the nine triples

$$\Delta(00), \Delta(01), \Delta(02), \Delta(10), \Delta(11), \Delta(12), \Delta(20), \Delta(21), \Delta(22),$$

namely the 27 numbers

$$1, 1, 5, 1, 3, 5, 3, 1, 5, 1, 1, 5, 1, 3, 5, 3, 1, 5, 1, 1, 5, 1, 3, 5, 3, 1, 5, 1, 1, 5, 1, 3, 5, 3, 1, 5.$$

We can write out the Stern Tri-atomic sequence via a recursive formula, which will take a bit to define. Start with three numbers a_1, a_2, a_3 , which for

us are usually the triple $1, 1, 1$. For each positive integer N , there is a unique tuple $I = (i_1, i_2, \dots, i_n)$, with each i_j a zero, one or two, and a k chosen to be as large as possible from $l \in \{1, 2, 3\}$ such that

$$\begin{aligned} N &= 3(1 + 3 + 3^2 + \dots + 3^{n-1}) + i_1 3^n + i_2 3^{K-1} + \dots + i_n 3 + k \\ &= \frac{3(3^k - 1)}{2} + i_1 3^n + i_2 3^{K-1} + \dots + i_n 3 + k. \end{aligned}$$

Define the function τ from positive integers to tuples $\times \{1, 2, 3\}$ by setting

$$\tau(N) = (I : k).$$

We want to write our eventual sequence as

$$\begin{aligned} &a_{(1)}, a_{(2)}, a_{(3)} \\ &a_{(0;1)}, a_{(0;2)}, a_{(0;3)}, a_{(1;1)}, a_{(1;2)}, a_{(1;3)}, a_{(2;1)}, a_{(2;2)}, a_{(2;3)} \\ &\vdots \end{aligned}$$

We define the following three functions

$$\tau_1 : \mathbb{N} \rightarrow \mathbb{N}, \tau_2 : \mathbb{N} \rightarrow \mathbb{N}, \tau_3 : \mathbb{N} \rightarrow \mathbb{N}$$

by setting

$$\begin{aligned} \tau_1(N) &= \tau_1(j_K, \dots, j_1; l) \\ &= (j_K, \dots, j_2, 1) \\ \tau_2(N) &= \tau_2(j_K, \dots, j_1; l) \\ &= (j_K, \dots, j_2, 2) \\ \tau_3(N) &= \tau_3(j_K, \dots, j_1; l) \\ &= (j_K, \dots, j_2, 3) \end{aligned}$$

Definition 3.3. *Given three numbers a_1, a_2 and a_3 , the corresponding Stern tri-atomic sequence is*

$$a_N = \begin{cases} a_{\tau_1(N)} & \text{if } j_1 = 0, l = 1 \\ a_{\tau_2(N)} & \text{if } j_1 = 0, l = 2 \\ a_{\tau_1(N)} + a_{\tau_2(N)} + a_{\tau_3(N)} & \text{if } j_1 = 0, l = 3 \\ a_{\tau_2(N)} & \text{if } j_1 = 1, l = 1 \\ a_{\tau_3(N)} & \text{if } j_1 = 1, l = 2 \\ a_{\tau_1(N)} + a_{\tau_2(N)} + a_{\tau_3(N)} & \text{if } j_1 = 1, l = 3 \\ a_{\tau_3(N)} & \text{if } j_1 = 2, l = 1 \\ a_{\tau_1(N)} & \text{if } j_1 = 2, l = 2 \\ a_{\tau_1(N)} + a_{\tau_2(N)} + a_{\tau_3(N)} & \text{if } j_1 = 2, l = 3 \end{cases}$$

Of course, we want to make sure that this definition agrees with our earlier one. Luckily the following proposition is a straightforward calculation.

Proposition 3.4. *We have for $\tau(N) = (I; k)$ that*

$$a_N = v_l(k).$$

The proof is just an unraveling of the definitions.

We also have a description for the series $\sum a_N x^N$. Suppose we have a triangle with vertices v_1, v_2, v_3 , which we write as column vectors and put into the matrix

$$V = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}.$$

Set

$$P(x) = A_0 + xA_1 + x^2A_2.$$

Let e_i be the standard three by one column vector with entries all zero save in the i th entry, which is one. When

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

then

$$\sum a_N x^N = e_3^T \cdot V \cdot \left[I + \sum_{k=1}^{\infty} x^{\frac{3(3^k-1)}{2}} P(x^{3^k}) P(x^{3^{k-1}}) \cdots P(x^3) \right] (xe_1 + x^2e_2 + x^3e_3).$$

The proof is again an unraveling of the notation, as follows.

Start with considering:

$$\begin{aligned} P(x)P(x^3) &= (A_0 + A_1x + A_2x^2)(A_0 + A_1x^3 + A_2x^6) \\ &= A_0A_0 + A_1A_0x + A_2A_0x^2 + A_0A_1x^3 + A_1A_1x^4 + A_2A_1x^5 \\ &\quad A_0A_2x^6 + A_1A_2x^7 + A_2A_2x^8 \\ P(x)P(x^3)P(x^9) &= (A_0 + A_1x + A_2x^2)(A_0 + A_1x^3 + A_2x^6)(A_0 + A_1x^9 + A_2x^{18}) \\ &= A_0A_0A_0 + A_1A_0A_0x + A_2A_0A_0x^2 + A_0A_1A_0x^3 + A_1A_1A_0x^4 \\ &\quad + A_2A_1A_0x^5 + A_0A_2A_0x^6 + A_1A_2A_0x^7 + A_2A_0A_2x^8 \\ &= A_0A_0A_1x^9 + A_1A_0A_1x^{10} + A_2A_0A_1x^{11} + A_0A_1A_1x^{12} \\ &\quad + A_1A_1A_1x^{13} + A_2A_1A_1x^{14} + A_0A_2A_1x^{15} + A_1A_2A_1x^{16} \\ &\quad + A_2A_2A_1x^{17} \\ &\quad + A_0A_0A_2x^{18} + A_1A_0A_2x^{19} + A_2A_0A_2x^{20} + A_0A_1A_2x^{21} \\ &\quad + A_1A_1A_2x^{22} + A_2A_1A_2x^{23} + A_0A_2A_2x^{24} + A_1A_2A_2x^{25} \\ &\quad + A_2A_2A_2x^{26} \\ &\quad \vdots \end{aligned}$$

Multiplying any three-by-three matrix by e_3^T on the left picks out the bottom row of the matrix. Multiplying any one-by-three matrix on the right by e_j picks out the j th term.

Putting all of this together gives us the result.

4 The Farey Map

4.1 Definition

Stern's di-atomic sequences are linked to continued fractions [12]. There is a multidimensional continued fraction algorithm which generates in an analogous fashion Stern's tri-atomic sequence. In [2], a (seemingly) new multi-dimensional continued fraction algorithm was given. Though the goal of that paper was to find a generalization of the Minkowski $\varphi(x)$ function, it is an algorithm that seems particularly well-suited to generalize Stern's diatomic sequence. (In [13], Panti used the Monkmeyer algorithm to generalize the Minkowski $\varphi(x)$ function.)

Let

$$\Delta = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\}.$$

Partition Δ into the three subtriangles

$$\Delta_0 = \{(x, y) \in \Delta : 1 - 2y \geq x - y \geq y\}$$

$$\Delta_1 = \{(x, y) \in \Delta : 2x - 1 \geq y \geq 1 - x\}$$

$$\Delta_2 = \{(x, y) \in \Delta : 1 - 2x + 2y \geq 1 - x \geq x - y\}.$$

The Farey map $T : \Delta \rightarrow \Delta$ is given by three one-to-one onto maps

$$T_i : \Delta_i \rightarrow \Delta$$

defined via

$$\begin{aligned} T_0(x, y) &= \left(\frac{x - y}{1 - 2y}, \frac{y}{1 - 2y} \right) \\ T_1(x, y) &= \left(\frac{y}{2x - 1}, \frac{1 - x}{2x - 1} \right) \\ T_2(x, y) &= \left(\frac{1 - x}{1 - 2x + 2y}, \frac{x - y}{1 - 2x + 2y} \right) \end{aligned}$$

These maps are the Farey analog of the Gauss map for continued fractions.

4.2 Farey Map on Vertices

Let \triangle be a triangle with three vertices v_1, v_2, v_3 , each written as a column vector in \mathbb{R}^3 . Subdivide \triangle into three sub-triangles

$$\begin{array}{ll} \triangle(0) & \text{with vertices } v_1, v_2, v_1 + v_2 + v_3 \\ \triangle(1) & \text{with vertices } v_2, v_3, v_1 + v_2 + v_3 \\ \triangle(3) & \text{with vertices } v_3, v_1, v_1 + v_2 + v_3 \end{array}$$

We have three maps

$$A_i : \triangle \rightarrow \triangle_i,$$

which we describe via matrix multiplication on the right.

Thus we set

$$A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

For any three vectors v_1 , v_2 and v_3 , we have

$$(v_1, v_2, v_3)A_0 = (v_1, v_2, v_1 + v_2 + v_3)$$

$$(v_1, v_2, v_3)A_1 = (v_2, v_3, v_1 + v_2 + v_3)$$

$$(v_1, v_2, v_3)A_2 = (v_3, v_1, v_1 + v_2 + v_3).$$

We now specify our initial three vectors, by setting

$$M = (v_1, v_2, v_3) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

This is simply the Farey map's analog of the Farey subdivision

5 Combinatorial Properties of Tri-Atomic Sequences

We now develop the some analogs of the combinatorial properties for Stern diatomic sequences for triatomic sequences.

5.1 Three-fold symmetry

Property 4 in Lehmer [11] is that there is a two fold symmetry for Stern diatomic sequences on each “line”. We will see that for tri-atomic sequences, we have a three-fold symmetry.

Proposition 5.1. *For any $N = (0, j_1, \dots, j_{K-1}, l)$, for $j_m = 0, 1$ or 2 and l is as large as possible for $l \in \{1, 2, 3\}$, we have*

$$a_N = a_{N+3^k} = a_{N+2 \cdot 3^N}.$$

Thus

$$a_{(0, j_1, \dots, j_{K-1}, l)} = a_{(1, j_1, \dots, j_{K-1}, l)} = a_{(2, j_1, \dots, j_{K-1}, l)}.$$

Proof. At the 0th level, we have

$$a_1 = 1, a_2 = 1, a_3 = 1.$$

The 1th level is

$$a_4, \dots, a_{12},$$

which by definition is

$$a_1, a_2, a_1 + a_2 + a_3, a_2, a_3, a_1 + a_2 + a_3, a_3, a_1, a_1 + a_2 + a_3,$$

which in turn is simply

$$1, 1, 3, 1, 1, 3, 1, 1, 3,$$

satisfying the desired relation.

The rest follows from the basic recursion relation.

□

5.2 Tribanacci numbers

Recall that tribanacci numbers are the terms in the sequence β_1, β_2, \dots given by the recursion relation

$$\beta_{k+3} = \beta_k + \beta_{k+1} + \beta_{k+2}.$$

Such sequences are intimately intertwined with tri-atomic sequences. Recall $\text{level}(N) = \text{level}(j_K, \dots, j_1, l) = K$. We write $J = (j_1, \dots, j_K)$, allowing us to write $N = (J : l)$.

Theorem 5.2. *For any K -tuple J , the following subsequence is a tribanacci sequence:*

$$a_{(J:1)}, a_{(J:2)}, a_{(J:3)}, a_{(J,1:3)}, a_{(J,1,1:3)}, a_{(J,1,1,1:3)}, \dots$$

Proof. We start with the triple, say at level K ,

$$a_{(J:1)}, a_{(J:2)}, a_{(J:3)}.$$

Then we have on level $K + 1$ the triple $a_{(J,1:1)}, a_{(J,1:2)}, a_{(J,1:3)}$ being

$$a_{(J:2)}, a_{(J:3)}, a_{(J:1)} + a_{(J:2)} + a_{(J:3)}.$$

On level $K + 2$, we have that the triple $a_{(J,1,1:1)}, a_{(J,1,1:2)}, a_{(J,1,1:3)}$ being

$$a_{(J:3)}, a_{(J,1:3)}, a_{(J:2)} + a_{(J:3)} + a_{(J,1:3)}.$$

This process continues, since by definition we have

$$a_{(J,1^s:3)} = a_{(J,1^{s-3}:3)} + a_{(J,1^{s-2}:3)} + a_{(J,1^{s-1}:3)},$$

giving us our result. □

Corollary 5.3. *The standard tribonacci sequence*

$$1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, \dots$$

is

$$a_1, a_2, a_3, a_{(1:3)}, a_{(1,1:3)}, \dots, a_{(1^s:3)}, \dots$$

5.3 Sums at Levels

In the traditional Stern diatomic sequence, Stern showed that the sum of all terms of level n is $3^n + 1$ (see property 2 in [11]). A similar sum exists for the Stern tri-atomic sequence.

Proposition 5.4. *The sum of all terms in the Stern tri-atomic sequence at the k th level is $5^k \cdot 3$. Thus for fixed k*

$$\sum a_{(J:l)} = 5^k \cdot 3,$$

where the summation is over all k -tuples $J = (j_1, \dots, j_k)$, for $j_m = 0, 1$ or 2 , and over $l = 1, 2, 3$.

Proof. The 0th level is the three numbers 1, 1, 1, and hence the sum is 3. Now suppose we have at the k level the number $N = (j_k, \dots, j_1 : 1)$. Then the triple

of numbers

$$a_N, a_{N+1}, a_{N+2}$$

is in the k th level. What is important is that this triple generates the following nine terms in the $(k+1)$ st level:

$$a_N, a_{N+1}, a_N + a_{N+1} + a_{N+2},$$

$$a_{N+1}, a_{N+2}, a_N + a_{N+1} + a_{N+2}$$

and

$$a_{N+2}, a_N, a_N + a_{N+1} + a_{N+2},$$

whose sum is

$$5(a_N + a_{N+1} + a_{N+2}),$$

giving us our result.

□

5.4 Values at Levels

We can write each positive integer $N = (K : j_1, \dots, j_K : l)$, where each $j_i \in \{0, 1, 2\}$ and each $l \in \{1, 2, 3\}$. We have said that the corresponding a_N is at the k th level. We set

$$\delta_K(n) = \text{number of times } n \text{ occurs in the } K\text{th level.}$$

We further set

$$\delta_K^1(n) = \text{number of times } n \text{ occurs in the } K\text{th level when } l = 1,$$

$$\delta_K^2(n) = \text{number of times } n \text{ occurs in the } K\text{th level when } l = 2,$$

and

$\delta_K^3(n)$ = number of times n occurs in the the K th level when $l = 3$.

Thus at level 1, our sequence is

$$1, 1, 3, 1, 1, 3, 1, 1, 3,$$

in which case

$$\delta_1(1) = 6, \delta_1(3) = 3, \delta_1(n) = 0 \text{ for } n \neq 1, 3,$$

$$\delta_1^1(1) = 3, \delta_1^1(3) = 1, \delta_1^1(n) = 0 \text{ for } n \neq 1, 3,$$

$$\delta_1^2(1) = 3, \delta_1^2(3) = 1, \delta_1^2(n) = 0 \text{ for } n \neq 1, 3,$$

$$\delta_1^3(1) = 0, \delta_1^3(3) = 3, \delta_1^3(n) = 0 \text{ for } n \neq 1, 3.$$

We now discuss some formulas for these new functions.

First for a technical lemma:

Lemma 5.5. *Suppose we are at the K th level, for some $N = (j_1, \dots, j_K; 1)$.*

Then the triple

$$a_N, a_{N+1}, a_{N+2}$$

will induce the following three triples on the $(K + 1)$ st level: The triple corresponding to

$$a_{(j_1, \dots, j_K, 0; 1)}, a_{(j_1, \dots, j_K, 0; 2)}, a_{(j_1, \dots, j_K, 0; 3)}$$

will be

$$(a_N, a_{N+1}, a_N + a_{N+1} + a_{N+2}),$$

the triple corresponding to

$$a_{(j_1, \dots, j_K, 1; 1)}, a_{(j_1, \dots, j_K, 1; 2)}, a_{(j_1, \dots, j_K, 1; 3)}$$

will be

$$(a_{N+1}, a_{N+2}, a_N + a_{N+1} + a_{N+2}),$$

and the triple corresponding to

$$a_{(j_1, \dots, j_K, 2; 1)}, a_{(j_1, \dots, j_K, 2; 2)}, a_{(j_1, \dots, j_K, 2; 3)}$$

will be

$$(a_{N+2}, a_N, a_N + a_{N+1} + a_{N+2}).$$

We will say we have a triple a_N, a_{N+1}, a_{N+2} at level K if $N = (j_K, \dots, j_1; 1)$.

Proposition 5.6.

$$\delta_K(2n) = \delta_K^1(2n) = \delta_K^2(2n) = \delta_K^3(2n) = 0.$$

Proof. If $N = (j_1, \dots, j_K; 1)$ at the K th level and if each term in the triple a_N, a_{N+1}, a_{N+2} is odd, then all of their descendants in the next level will still be odd, from the above lemma. Thus to prove the proposition, we just have to observe that at the initial level 0, our numbers are 1, 1, 1, forcing all subsequent elements to be odd. \square

Proposition 5.7. *For any odd number $2n + 1 > 1$, we have*

$$\delta_n^3(2n + 1) = 9.$$

Proof. We first show that if $N = (0^n; 1)$, then the corresponding triple is $(1, 1, 2n + 1)$. The proof is by induction. At the initial zero level, the initial triple is $(1, 1, 1)$ as desired. Then suppose at the n th level the triple corresponding to $N = (0^n; 1)$ is the desired $(1, 1, 2n + 1)$. Then we have at the next level three triples descending from $(1, 1, 2n + 1)$, namely $(1, 1, 2n + 3)$, $(1, 2n + 1, 2n + 3)$ and $(2n + 1, 1, 2n + 3)$. By the three-fold symmetry of the sequence, this means that

$$\delta_{n+1}(2n + 3) \geq 9.$$

Now to show that this number cannot be greater than 9. Suppose at level $n \geq 1$ there are only three triples of the form $(1, 1, 2n + 1)$. This is true at level one. For any triple (a_N, a_{N+1}, a_{N+2}) at level n with a_N or a_{N+1} greater than one, then none of this triple's descendants will contribute to $\delta_{n+1}^3(2n + 3)$. Then at the next level, we will only pick up three triples $(1, 1, 2n + 3)$. Putting all this together, means that the only triples at level n that will have descendants contributing to $\delta_{n+1}^3(2n + 3)$ must be of the form $(1, 1, 2n + 1)$, and each of these three triples will contribute three $(2n + 3)$ s. Thus we have our result.

□

Proposition 5.8.

$$\delta_k(m) = \delta_{k+1}^1(m) = \delta_{k+1}^2(m).$$

Proof. This follows immediately from the construction of the sequence. □

Proposition 5.9.

$$\delta_{k+m}(2k + 1) = 2^m \delta_k(2k + 1).$$

Proof. It is at the k th level where $2k + 1$ can last occur in a δ^3 term. Hence this follows from the previous proposition, since we have

$$\begin{aligned}
 \delta_{k+m}(2k+1) &= \delta_{k+m}^1(2k+1) + \delta_{k+m}^2(2k+1) + \delta_{k+m}^3(2k+1) \\
 &= \delta_{k+m-1}(2k+1) + \delta_{k+m-1}(2k+1) + 0 \\
 &= 2\delta_{k+m-1}(2k+1).
 \end{aligned}$$

□

Proposition 5.10.

$$\delta_k(2k+1) = 9 + 2\delta_{k-1}(2k+1)$$

Proof. We have already seen that $\delta_k^3(2k+1) = 9$. Since

$$\delta_k(2k+1) = \delta_k^1(2k+1) + \delta_k^2(2k+1) + \delta_k^3(2k+1) = \delta_k^1(2k+1) + \delta_k^2(2k+1) + 9,$$

all we need show is that $\delta_k^1(2k+1) + \delta_k^2(2k+1) = 2\delta_{k-1}(2k+1)$. But this follows from the fact that each triple a, b, c at the $k - 1$ st level generates for the k th level the triples $a, b, a + b + a$, or $b, c, a + b + c$, or $c, a, a + b + c$ and each number appearing in the $k - 1$ st level appears twice in the k th level.

□

Of course, it is easy to generate a table of various $\delta_K(n)$. The following is a

list of some examples.

(K, n)	$\delta_K(n)$	$\delta_K^1(n)$	$\delta_K^2(n)$	$\delta_K^3(n)$
(0, 1)	3	1	1	1
(1, 1)	6	3	3	0
(1, 3)	3	0	0	3
(2, 1)	12	6	6	0
(2, 3)	6	3	3	0
(2, 5)	9	0	0	9
(3, 1)	24	12	12	0
(3, 1)	24	12	12	0
(3, 3)	12	6	6	0
(3, 5)	18	9	9	0
(3, 7)	9	0	0	9
(3, 9)	18	0	0	18
(4, 1)	48	24	24	0
(4, 3)	24	12	12	0
(4, 5)	36	18	18	0
(4, 7)	18	9	9	0
(4, 9)	45	18	18	9
(4, 13)	36	0	0	36
(4, 15)	18	0	0	18
(4, 17)	18	0	0	18
(5, 1)	96	48	48	0
(5, 3)	48	24	24	0
(5, 5)	72	36	36	0
(5, 7)	36	18	18	0
(5, 9)	90	45	45	0
(5, 11)	9	0	0	9
(5, 13)	72	36	36	0
(5, 15)	36	18	18	0
(5, 17)	72	18	18	36
(5, 19)	18	0	0	18
(5, 21)	18	0	0	18
(5, 23)	18	0	0	18
(5, 25)	72	0	0	72
(5, 29)	36	0	0	36
(5, 31)	18	0	0	18

6 Paths of a directed graph

There is a simple interpretation of the Stern tri-atomic sequence as the number of paths in a directed graph from three initial vertices. Recall the

diagrams in section. Starting with a triangle \triangle with vertices v_1, v_2, v_3 , we systematically constructed a subdivision of \triangle so that at level K , we have 3^K triangles. It is quite easy to put the structure of a directed graph on this subdivision. Suppose we have one of the subtriangles $\triangle(I)$, where recall I is a K -tuple of 0, 1 or 2. Its vertices are denoted by $v_1(I), v_2(I)$ and $v_3(I)$. Then for level $K + 1$, we add one more vertex, denoted by $v_1(I) + v_2(I) + v_3(I)$ and three directed paths, one from each $v_1(I)$ to the new vertex. Finally, note that there are no direct paths between any of the three initial vertices. Recall our earlier notation of writing any positive integer N as $N = (j_1, j_2, \dots, j_K; l)$ if

$$N = 3(1 + 3 + 3^2 + \dots + 3^{K-1}) + j_1 3^K + j_2 3^{K-1} + \dots + j_K 3 + l,$$

where each j_i is zero, one or two and l is chosen to be as large as possible from the set $\{1, 2, 3\}$.

Theorem 6.1. *For any positive integer N , the number a_N is the number of paths from v_1, v_2 and v_3 to the l th vertex of the $\triangle(j_1, \dots, j_K)$, save for when the corresponding vertex of $\triangle(j_K, \dots, j_1)$ is one of the initial v_1, v_2 or v_3 .*

The proof is straightforward.

7 Questions

The Farey map is only one type of multidimensional continued fraction. (For background on multi-dimensional continued fractions, see Schweiger [45]). Recently, in [3], a family of multi-dimensional continued fractions was created which includes many previously well-known algorithms, though not the Farey

map. In [4], analogs of Stern's diatomic sequence are studied. Independently, Goldberg [7] has started finding analogs for the Monkemayer map. It would also be interesting to find the Stern analogs in the language of Lagarias [10].

Also, there are many properties of Stern's diatomic sequences whose analogs for the Farey map have yet to be discovered. Probably the most interesting would be to find the analog of the link between Stern's sequence and the number of hyperbinary representations there are for positive integers, as discussed in [12].

Another interesting problem is to attempt a multidimensional analog of the recently found link between the Tower of Hanoi graph and Stern's diatomic sequence by Hinz, Klavžar, Milutinović, Parisse and Petr [8].

Finally, it would be interesting to extend the polynomial analogs of Stern's diatomic sequence (as in the work of Dilcher and Stokarsky [5], [6], of Klavžar, Milutinović and Petr [9], of Ulas and Ulas [20], of Vargas [21], and of Allouche and Mendès France [1]) to finding polynomial analogs for triatomic sequences.

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